

Conformal symmetry and non-relativistic second order fluid dynamics

Jingyi Chao and Thomas Schäfer

Physics Department,

North Carolina State University,

Raleigh, NC 27695, USA

(Dated: August 26, 2011)

Abstract

We study the constraints imposed by conformal symmetry on the equations of fluid dynamics at second order in gradients of the hydrodynamic variables. At zeroth order conformal symmetry implies a constraint on the equation of state, $\mathcal{E}_0 = \frac{2}{3}P$, where \mathcal{E}_0 is the energy density and P is the pressure. At first order, conformal symmetry implies that the bulk viscosity must vanish. We show that at second order conformal invariance requires that two-derivative terms in the stress tensor must be traceless, and that it determines the relaxation of dissipative stresses to the Navier-Stokes form. We verify these results by solving the Boltzmann equation at second order in the gradient expansion. We find that only a subset of the terms allowed by conformal symmetry appear.

I. INTRODUCTION

There has been great interest recently in the dynamics of conformally invariant fluids. One motivation is the observation that the AdS/CFT correspondence [1], which relates the dynamics of a four-dimensional conformal field theory to quantum gravity in ten dimensions, also implies a fluid-gravity correspondence [2–4]. The fluid-gravity correspondence relates solutions of the Navier-Stokes equation for conformally invariant fluids in four dimensions to dynamical black hole solutions of the Einstein equations in higher dimensions. The fluid-gravity correspondence has been used to establish the form of second order terms in the equations of relativistic fluid dynamics [3, 5]. The correspondence has also been extended to certain non-conformal fluids [6], superfluids [7, 8], and to the incompressible non-relativistic Navier-Stokes equation [9].

Another motivation comes from the experimental discovery of nearly perfect fluidity in certain conformal or almost conformal fluids [10]. The first example is the observation of nearly ideal flow of the quark gluon plasma produced in heavy ion collisions at the relativistic heavy ion collider (RHIC) [11]. This discovery was soon followed by the discovery of nearly ideal fluid dynamics in a very different system, the dilute Fermi gas at unitarity [12]. Both the quark gluon plasma and the dilute Fermi gas are characterized by a very small shear viscosity to entropy density ratio, $\eta/s \lesssim 0.4\hbar/k_B$ [13–19]. This result is close to the value $\eta/s = \hbar/(4\pi k_B)$ obtained in the strong coupling limit of a super-conformal Yang Mills plasma [20, 21].

In the present work we will focus on the dilute Fermi gas and study the hydrodynamics of a scale invariant non-relativistic fluid. At unitarity the two-body scattering length of the fermions is infinite and the range of the interaction is zero. This implies that the system is exactly scale invariant [22–24]. Hydrodynamics is an effective theory that governs the behavior of the system at long distances. The equations of motion are organized as an expansion in the number of derivatives acting on the hydrodynamic variables. In the case of a one-component non-relativistic fluid the hydrodynamic variables are the temperature T , the density n , and the fluid velocity \vec{V} . At zeroth order in the gradient expansion conformal symmetry implies a constraint on the equation of state, $\mathcal{E}_0 = \frac{2}{3}P$, where $\mathcal{E}_0(n, T)$ is the energy density and $P(n, T)$ is the pressure. At first order, conformal symmetry implies that the contribution of the divergence of the velocity field to the pressure, $\delta P = \zeta(\nabla \cdot V)$, must

vanish [25–27]. This means that the bulk viscosity ζ of a scale invariant fluid is zero. In this work we will study constraints from conformality at second order in the gradient expansion.

Second order hydrodynamics has a long history [28–30] but the theory is rarely used in practice. An important exception is the relativistic case, where the second order theory has the advantage of being manifestly causal [31]. There is a similar reason for using second order hydrodynamics in certain non-relativistic flows. The expansion of a scale invariant fluid after the release from a harmonic trap leads to a Hubble-like flow in which the expansion velocity is linear in the coordinates [18]. This implies that ideal stresses propagate outward with a finite velocity, but at first order in gradients dissipative stresses build up instantaneously everywhere in space. This unphysical behavior can be avoided by taking into account a finite relaxation time for the dissipative stresses. We will show that relaxation time effects appear naturally at second order in the gradient expansion.

There are additional reasons for studying higher order terms in the equations of fluid dynamics. Higher order terms provide information about the convergence of the gradient expansion, and they lead to new Kubo relations. In the relativistic case it is known that some of these Kubo formulas relate transport properties to thermodynamic quantities [32]. Finally, understanding the structure of the gradient expansion may help in constructing holographic duals of non-relativistic conformal field theories [33–35]. The paper is structured as follows. In Sect. II we review a local extension of Galilean and conformal invariance and study constraints on the structure of gradient terms in fluid dynamics. In Sect. III we consider a specific kinetic equation and solve for the stress tensor and the entropy current at second order in gradients. We show that the result agrees with the general constraints, but that the kinetic theory only generates a subset of the allowed terms. We conclude with a discussion of open questions.

II. CONFORMAL HYDRODYNAMICS

A. Navier-Stokes equation

The constraints imposed by Galilean invariance and conformal symmetry on the structure of the Navier-Stokes equation were studied by Son [25] based on earlier work by Son and Wingate [22]. The basic strategy is to generalize Galilean and scale invariance to

local symmetries. For this purpose we consider the fluid moving in a curved background characterized by the metric $g_{ij}(x, t)$. We also include background $U(1)$ gauge fields $A_0(x, t)$ and $A_i(x, t)$. The Navier-Stokes equations in curved space are

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}\rho) + \nabla_i(\rho V^i) = 0, \quad (1)$$

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}\rho V_i) + \nabla_k \Pi_i^k = \frac{\rho}{m} Q_i, \quad (2)$$

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}s) + \nabla_k J_s^k = \frac{R}{T}, \quad (3)$$

where $g = \det(g_{ij})$ and ∇_i is the covariant derivative associated with the metric g_{ij} . The hydrodynamic variables are the mass density ρ , the entropy density s , and the fluid velocity V^i . The energy momentum tensor is $\Pi_{ij} = \Pi_{ij}^0 + \delta\Pi_{ij}$, where

$$\Pi_{ij}^0 = \rho V_i V_j + P g_{ij} \quad (4)$$

is the ideal fluid part, and $\delta\Pi_{ij}$ is the viscous correction. The entropy current is $J_s^i = s V^i + \delta J_s^i$, and R is the dissipative function. We will specify the viscous terms $\delta\Pi_{ij}$, δJ_s^i and R below. The Lorentz force Q_i is defined as $Q_i = E_i - F_{ij} V^j$ with $E_i = \partial_t A_i - \partial_i A_0$ and $F_{ij} = \partial_i A_j - \partial_j A_i$.

B. Diffeomorphism invariance

The equations of fluid dynamics are invariant under a combination of local diffeomorphisms $x^i \rightarrow x^i + \xi^i(x, t)$ and gauge transformations $A_0 \rightarrow A_0 - \dot{\alpha}$, $A_i \rightarrow A_i - \partial_i \alpha$. This symmetry generalizes Galilean invariance. A Galilei transformation with boost velocity U^i corresponds to a flat background metric $g_{ij} = \delta_{ij}$ and $\xi^i = U^i t$ as well as $\alpha = m U^i x^i$. The hydrodynamic variables are invariant under gauge transformations. They transform under diffeomorphisms as

$$\delta\rho = -\xi^k \nabla_k \rho, \quad (5)$$

$$\delta s = -\xi^k \nabla_k s, \quad (6)$$

$$\delta V^i = -\xi^k \nabla_k V^i + V^k \nabla_k \xi^i + \dot{\xi}^i. \quad (7)$$

We note that ρ and s transform as scalars, but V_i does not transform as a vector. We have $\delta V^i = \mathcal{L}_\xi V^i + \dot{\xi}^i$, where \mathcal{L}_ξ is the Lie derivative and $\dot{\xi}^i$ is an “extra” term. The gauge fields

and the metric transform as

$$\delta A_0 = -\xi^k \nabla_k A_0 - A_k \dot{\xi}^k, \quad (8)$$

$$\delta A_i = -\xi^k \nabla_k A_i - A_k \nabla_i \xi^k + m g_{ik} \dot{\xi}^k, \quad (9)$$

$$\delta g_{ij} = -g_{ik} \nabla_j \xi^k - g_{kj} \nabla_i \xi^k. \quad (10)$$

Using the transformation laws for the gauge fields we can deduce the transformation properties of the field strengths and the Lorentz force. We find

$$\delta E_i = \mathcal{L}_\xi E_i + F_{ik} \dot{\xi}^k + m \dot{g}_{ik} \xi^k + m g_{ik} \ddot{\xi}^k, \quad (11)$$

$$\delta F_{ij} = \mathcal{L}_\xi F_{ij} + m g_{jk} \nabla_i \dot{\xi}^k - m g_{ik} \nabla_j \dot{\xi}^k, \quad (12)$$

$$\delta Q_i = \mathcal{L}_\xi Q_i + m \left(\dot{g}_{ik} \xi^k + g_{ik} \ddot{\xi}^k + V_k \nabla_i \dot{\xi}^k - g_{ik} V^j \nabla_j \dot{\xi}^k \right). \quad (13)$$

Using these ingredients we can verify that the hydrodynamic equations (1-3) are invariant. We note that under diffeomorphisms the ideal parts of the momentum and entropy currents do not transform as vectors, and the ideal part of the stress tensor does not transform as a tensor. The extra terms in the transformation laws are canceled by terms that arise from the time derivatives of the conserved charges and the Lorentz force. In order to maintain diffeomorphism invariance beyond the level of ideal hydrodynamics dissipative corrections to the entropy current and the stress tensor have to transform as vectors and second rank tensors, respectively.

At first order in gradients dissipative terms involve the heat current and the shear and bulk stresses [36]. The heat current

$$q_i = -\nabla_i \log(T), \quad (14)$$

where ∇_i is the covariant derivative, transforms as a vector under diffeomorphisms. The shear and bulk stresses can be promoted to second rank tensors by adding extra terms involving \dot{g}_{ij} [25]. We will show below that these terms automatically appear in kinetic theory. We have

$$\sigma_{ij} = \nabla_i V_j + \nabla_j V_i + \dot{g}_{ij} - \frac{2}{3} g_{ij} \langle \sigma \rangle, \quad (15)$$

$$\langle \sigma \rangle = \nabla \cdot V + \frac{\dot{g}}{2g} \quad (16)$$

where σ_{ij} is the shear stress tensor and $g_{ij} \langle \sigma \rangle$ is the bulk stress tensor.

C. Conformal invariance

Conformal transformations are infinitesimal rescalings of the time variable, $t \rightarrow t + \beta(t)$. A non-relativistic scale transformation corresponds to $\beta = bt$ combined with a diffeomorphism generated by $\xi^k = \frac{1}{2}bx^k$. Special conformal transformations are generated by $\beta = -ct^2$, $\xi^k = -ctx^k$ and $\alpha = -\frac{1}{2}mcx^2$. An operator O is said to have conformal dimension $\Delta_O = [O]$ if

$$\delta O = -\beta \dot{O} - \Delta_O \dot{\beta} O. \quad (17)$$

Examples of operators with well-defined conformal dimension are

$$[g_{ij}] = -1, \quad [g^{ij}] = +1, \quad [A_0] = +1, \quad [A_i] = 0. \quad (18)$$

We also have $[O_1 O_2] = [O_1] + [O_2]$ and $[\partial_i O] = [O]$, but $\partial_t O$ does not, in general, have a well-defined conformal dimension. The conformal dimensions of the hydrodynamic variables are

$$[\rho] = \frac{3}{2}, \quad [s] = \frac{3}{2}, \quad [V_i] = 0. \quad (19)$$

We can now check under what conditions the equations of hydrodynamics are conformally invariant. Two key relations are

$$\delta \dot{g}_{ij} = -\beta \ddot{g}_{ij} + \ddot{\beta} g_{ij} \equiv \delta^{\Delta=0}(\dot{g}_{ij}) + \ddot{\beta} g_{ij}, \quad (20)$$

and

$$\delta \left(\frac{\dot{g}}{2g} \right) = -\beta \partial_t \left(\frac{\dot{g}}{2g} \right) - \dot{\beta} \left(\frac{\dot{g}}{2g} \right) + \frac{3}{2} \ddot{\beta} = \delta^{\Delta=1} \left(\frac{\dot{g}}{2g} \right) + \frac{3}{2} \ddot{\beta}, \quad (21)$$

where $\delta^{\Delta=d} O$ denotes the transformation of a conformal operator O of dimension d . Using equ. (20) and (21) we can show that the equations of fluid dynamics are conformally invariant if

$$[\Pi_{ij}] = \frac{3}{2}, \quad [(j_s)_i] = \frac{3}{2}, \quad [R] = \frac{7}{2}. \quad (22)$$

These relations are satisfied in the case of ideal fluid dynamics (in ideal hydrodynamics $R = 0$). At one derivative order we find $[\sigma_{ij}] = 0$, $[q_i] = 0$ and $\delta \langle \sigma \rangle = \delta^{\Delta=1} \langle \sigma \rangle + \frac{3}{2} \ddot{\beta}$. This implies that

$$\delta \Pi_{ij} = -\eta \sigma_{ij}, \quad \delta (j_s)_i = \kappa q_i, \quad (23)$$

are conformally invariant provided $[\eta] = [\kappa] = \frac{3}{2}$, but $\delta \Pi_{ij} = -\zeta g_{ij} \langle \sigma \rangle$ violates conformal invariance unless $\zeta = 0$. We note that the conformal dimension of η and κ agree with their naive scaling dimension.

D. Second order terms: Building blocks

In order to construct the most general second order terms we begin by listing conformally invariant building blocks, that is diffeomorphism invariant scalars, vectors, and tensors with well-defined conformal dimensions. These objects are constructed from the hydrodynamic variables T, P and V^i , the metric g_{ij} and the gauge fields A_0, A_i , and time or spatial derivatives. Instead of T and P it is sometimes useful to consider T and the dimensionless variable $\alpha = \mu/T$, where μ is the chemical potential, as independent quantities. We begin with one-derivative scalars. Diffeomorphism invariant one-derivative scalars constructed from the hydrodynamic variables are DT , DP and $\langle\sigma\rangle$, where $D = \partial_0 + V^k \nabla_k$ is the comoving derivative. Because there are two scalar equations of motion only one of these objects is linearly independent. Neither one of the one-derivative scalars has a well-defined conformal dimension but we can construct conformal scalars by taking linear combinations. The quantity

$$\mathcal{S} = \left(D + \frac{2}{3} \langle\sigma\rangle \right) T \quad (24)$$

is a conformal scalar with dimension $[\mathcal{S}] = 2$. We will show, however, that at leading order in the derivative expansion \mathcal{S} vanishes by the equations of motion. We can construct three conformal vectors

$$\mathcal{V}_i^1 = \nabla_i T, \quad \mathcal{V}_i^2 = \nabla_i P, \quad \mathcal{V}_i^3 = DV_i - \frac{Q_i}{m}, \quad (25)$$

with weights $[\mathcal{V}_i^1] = 1$, $[\mathcal{V}_i^2] = \frac{5}{2}$, and $[\mathcal{V}_i^3] = 1$. There is one vector equation of motion and, up to higher order terms in the derivative expansion, \mathcal{V}^3 can be expressed in terms of \mathcal{V}^2 . There are two one-derivative tensors. The first is the shear tensor defined in equ. (15). The second is related to the vorticity of the fluid. We have

$$\mathcal{T}_{ij}^1 = \sigma_{ij}, \quad \mathcal{T}_{ij}^2 \equiv \Omega_{ij} = \nabla_i V_j - \nabla_j V_i - \frac{F_{ij}}{m}, \quad (26)$$

with $[\mathcal{T}_{ij}^1] = [\mathcal{T}_{ij}^2] = 0$.

E. Second order terms: Stress tensor

In this section we list conformal two-derivative tensors that can contribute to the stress tensor in a conformal theory. Rotational invariance implies that the stress tensor has to be symmetric. We will argue below that in a conformal theory the dissipative part of the stress

tensor also has to be traceless. One class of two-derivative tensors arises from contracting two one-derivative tensors. We find

$$\mathcal{O}_{ij}^1 = \sigma_{\langle i}{}^k \sigma_{j\rangle k}, \quad \mathcal{O}_{ij}^2 = \sigma_{\langle i}{}^k \Omega_{j\rangle k}, \quad \mathcal{O}_{ij}^3 = \Omega_{\langle i}{}^k \Omega_{j\rangle k}. \quad (27)$$

with $[\mathcal{O}_{ij}^1] = [\mathcal{O}_{ij}^2] = [\mathcal{O}_{ij}^3] = 1$. Here, $\mathcal{O}_{\langle ij\rangle} = \frac{1}{2}(\mathcal{O}_{ij} + \mathcal{O}_{ji} - \frac{2}{3}g_{ij}\mathcal{O}^k{}_k)$ denotes the symmetric traceless part of \mathcal{O}_{ij} . There is a unique diffeomorphism invariant tensor of conformal dimension +1 that can be constructed from the comoving derivative of σ_{ij} . This tensor is given by

$$\mathcal{O}_{ij}^4 = g_{ik}\dot{\sigma}_j^k + V^k\nabla_k\sigma_{ij} + \frac{2}{3}\langle\sigma\rangle\sigma_{ij} + F_{(i}{}^k\sigma_{kj}), \quad (28)$$

where $\mathcal{O}_{(ij)} = \frac{1}{2}(\mathcal{O}_{ij} + \mathcal{O}_{ji})$. There is a set of conformal tensors that can be obtained as the tensor product of two one-derivative tensors, or as two covariant derivatives acting on a scalar. We find

$$\mathcal{O}_{ij}^5 = \nabla_{\langle i}T\nabla_{j\rangle}T, \quad \mathcal{O}_{ij}^6 = \nabla_{\langle i}P\nabla_{j\rangle}P, \quad \mathcal{O}_{ij}^7 = \nabla_{\langle i}T\nabla_{j\rangle}P, \quad (29)$$

$$\mathcal{O}_{ij}^8 = \nabla_{\langle i}\nabla_{j\rangle}T, \quad \mathcal{O}_{ij}^9 = \nabla_{\langle i}\nabla_{j\rangle}P, \quad (30)$$

with $[\mathcal{O}_{ij}^{5-9}] = \{2, 5, \frac{7}{2}, 1, \frac{5}{2}\}$. Finally, in curved space there is a two-derivative tensor constructed only from the metric. This is the traceless part of the Ricci tensor,

$$\mathcal{O}_{ij}^{10} = R_{\langle ij\rangle}. \quad (31)$$

We can compare this list with the analogous result in the relativistic case [5]. In relativistic fluid dynamics there are five second order terms (four in flat space). Three of those are analogous to $\mathcal{O}_{ij}^1 - \mathcal{O}_{ij}^3$, and one is similar to \mathcal{O}_{ij}^4 . It is interesting to note that the relativistic analog of equ. (28) has a coefficient $\frac{1}{3}$ in front of $\langle\sigma\rangle$ instead of $\frac{2}{3}$. This difference is similar to the difference between the relativistic and non-relativistic conformal equation of state, $\mathcal{E}_0 = \frac{1}{3}P$ versus $\mathcal{E}_0 = \frac{2}{3}P$. In a relativistic theory without conserved charges there are only two hydrodynamic variables, T and the four-velocity U^μ , and as a result there are no terms analogous to $\mathcal{O}_{ij}^6, \mathcal{O}_{ij}^7$ and \mathcal{O}_{ij}^9 . Finally, in the relativistic case there is only one linear combination of \mathcal{O}_{ij}^5 and \mathcal{O}_{ij}^8 which is conformal, and that linear combination can be related to \mathcal{O}_{ij}^4 using the equations of motion. We can also compare our result to the structure of the Burnett equations [30]. The Burnett equations contain six independent two-derivative terms in Π_{ij} . The reason that the number of independent terms is smaller despite the fact that

there are no restrictions from conformal symmetry is that the Burnett equations are derived from kinetic theory, which only leads to a subset of the terms allowed by the symmetries.

We end this section by showing that there are no trace terms in $\delta\Pi_{ij}$, despite the fact that second order terms like $g_{ij}\sigma_{kl}\sigma^{kl}$ are conformal tensors. In Sec. III A we will show that in a non-trivial background the equation of energy conservation is given by

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}\mathcal{E}) + \nabla_k j_\epsilon^k = -\frac{1}{2}\dot{g}_{ij}\Pi^{ij}, \quad (32)$$

where $\mathcal{E} = \mathcal{E}_0 + \frac{1}{2}\rho V_i V^i$ is the energy density and $j_\epsilon^i = (\mathcal{E} + P)V^i + \delta j_\epsilon^i$ is the energy current. Equ. (32) is conformally invariant if

$$\mathcal{E}_0 = \frac{3}{2}P, \quad g^{ij}\delta\Pi_{ij} = 0, \quad (33)$$

which shows that the equation of state has to be conformal, and the dissipative part of the stress tensor must be traceless.

F. Second order terms: Entropy current

In this section we will use the results of Sec. II D to construct two-derivative conformal vectors that can contribute to the entropy current. There is one vector that can be written as the comoving derivative of a one-derivative vector

$$\mathcal{Q}_i^1 = Dq_i, \quad (34)$$

with $[\mathcal{Q}_i^1] = 1$. Note that $D\nabla_i\alpha$ is not an independent vector, because it can be related to \mathcal{Q}_i^1 and the vectors $\mathcal{Q}_i^{4,5}$ defined below using the hydrodynamic equation for $D\alpha$. We find four vectors that can be written as contractions of one-derivative vectors with one-derivative tensors

$$\mathcal{Q}_i^2 = q^j\sigma_{ij}, \quad \mathcal{Q}_i^3 = q^j\Omega_{ij}, \quad \mathcal{Q}_i^4 = (\nabla^j\alpha)\sigma_{ij}, \quad \mathcal{Q}_i^5 = (\nabla^j\alpha)\Omega_{ij}, \quad (35)$$

with $[\mathcal{Q}_i^{2-5}] = 1$. Finally, there are two vectors that can be constructed from the covariant derivative ∇_i acting on one-derivative tensors,

$$\mathcal{Q}_i^6 = \nabla^j\sigma_{ij}, \quad \mathcal{Q}_i^7 = \nabla^j\Omega_{ij}, \quad (36)$$

with $[\mathcal{Q}_i^{6,7}] = 1$. Again, the number of terms is bigger than the number of independent parameters in the entropy current in the Burnett equation, which is five.

III. KINETIC THEORY

A. Conservation laws

In this section we will investigate the structure of higher order terms starting from the Boltzmann equation. Our motivation is three-fold: i) We will verify that the results indeed satisfy the constraints derived in the previous section; ii) we will show that kinetic theory (within the relaxation time approximation) only leads to a subset of the allowed terms; iii) we will demonstrate that this subset manifestly satisfies the second law of thermodynamics.

The kinetic equation in a curved background $g_{ij}(x, t)$ can be found by starting from the Boltzmann equation in a four-dimensional curved space [37–39]

$$\frac{1}{p^0} \left(p^\mu \frac{\partial}{\partial x^\mu} - \Gamma_{\alpha\beta}^i p^\alpha p^\beta \frac{\partial}{\partial p^i} \right) f(t, x, p) = C[f], \quad (37)$$

where $f(t, x, p)$ is the distribution function, $C[f]$ is the collision term, $\Gamma_{\mu\nu}^\alpha$ is the Christoffel symbol associated with the four-dimensional covariant derivative ∇_μ , i, j, k are three-dimensional indices and μ, α, β are four-dimensional indices. In the non-relativistic limit [22]

$$\Gamma_{\alpha\beta}^i p^\alpha p^\beta = \Gamma_{jk}^i p^j p^k - \frac{1}{m} E^i p^0 p^0 + \frac{1}{m} g^{il} F_{lk} p^k p^0 + g^{il} \dot{g}_{lk} p^k p^0, \quad (38)$$

where $p^0 \simeq m$, E^i is the electric field and F_{lk} is the magnetic field introduced in Sect. II A. In the following we will drop the Lorentz force (these terms can always be restored using the symmetries described in Sect. II B). We get

$$\mathcal{D}f(t, x, p) = C[f], \quad (39)$$

where we have defined the Boltzmann operator

$$\mathcal{D} = \frac{\partial}{\partial t} + \frac{p^k}{m} \frac{\partial}{\partial x_k} - \frac{\Gamma_{jk}^i p^j p^k}{m} \frac{\partial}{\partial p^i} - g^{il} \dot{g}_{lk} p^k \frac{\partial}{\partial p^i}. \quad (40)$$

Consider a collision term that describes elastic two-body collisions $p_1 + p_2 \rightarrow p_3 + p_4$. The symmetries of the collision term imply that

$$\int d\Gamma \sqrt{g} \chi C[f] = \frac{1}{4} \int d\Gamma \sqrt{g} f C[\chi_1 + \chi_2 - \chi_{1'} - \chi_{2'}], \quad (41)$$

where $d\Gamma = d^3p$ and $\chi_i \equiv \chi(p_i)$. This implies that moments of the collision term with regard to the collision invariants $\chi = m, p^i, g_{ij} p^i p^j / (2m)$ must vanish. We will denote averages over

the momentum distribution as

$$\langle A \rangle = \int d\Gamma \sqrt{g} f A. \quad (42)$$

We define the conserved charges and currents as

$$\rho = \langle m \rangle, \quad \pi^k = \langle p^k \rangle, \quad \mathcal{E} = \frac{1}{2m} \langle g_{ij} p^i p^j \rangle, \quad (43)$$

$$\Pi^{ij} = \frac{1}{m} \langle p^i p^j \rangle, \quad j_\epsilon^i = \frac{1}{2m^2} \langle p^i g_{jk} p^j p^k \rangle. \quad (44)$$

The three collision invariants lead to three conservation laws. The first conservation law is obtained by taking a moment of the Boltzmann with $\chi = m$. We get

$$\begin{aligned} 0 &= \int d\Gamma \sqrt{g} m \mathcal{D}f(t, x, p) \\ &= \frac{\partial \langle m \rangle}{\partial t} - \frac{\dot{g}}{2g} \langle m \rangle + \partial_k \langle p^k \rangle - \frac{\partial_k g}{2g} \langle p^k \rangle - \frac{\partial}{\partial p^i} \langle \Gamma_{jk}^i p^j p^k + m g^{il} \dot{g}_{lk} p^k \rangle + \left\langle 2\Gamma_{ji}^i p^j + \frac{m \dot{g}}{g} \right\rangle \\ &= \frac{\partial_t (\sqrt{g} \rho)}{\sqrt{g}} + \nabla_k \pi^k. \end{aligned} \quad (45)$$

The momentum equation follows from taking a moment with $\chi = g_{ij} p^j$,

$$0 = \int d\Gamma \sqrt{g} g_{ij} p^j \mathcal{D}f(t, x, p) = \frac{\partial_t (\sqrt{g} \pi_i)}{\sqrt{g}} + \nabla_k \Pi^k_i. \quad (46)$$

and the energy follows from integrating over $\chi = g_{ij} p^i p^j / (2m)$,

$$0 = \frac{1}{2m} \int d\Gamma \sqrt{g} g_{ij} p^i p^j \mathcal{D}f(t, x, p) = \frac{\partial_t (\sqrt{g} \mathcal{E})}{\sqrt{g}} + \nabla_k j_\epsilon^k + \frac{1}{2} \dot{g}_{ij} \Pi^{ij}. \quad (47)$$

This is the result we have used in equ. (32) above. In order to determine the form of the currents in terms of the hydrodynamic variables we need to know the functional form of the distribution function near equilibrium. We will see shortly that $f(x, p, t) = f^{(0)}(g_{ij} c^i c^j)$ with $c_i(x, t) = v_i - V_i(x, t)$ and $v_i = p_i/m$. Then

$$\pi_i = \rho V_i, \quad \Pi_{ij} = \rho V_i V_j + P g_{ij}, \quad j_\epsilon^i = \left(\mathcal{E}_0 + P + \frac{1}{2} \rho V_j V^j \right) V^i, \quad (48)$$

with $\mathcal{E}_0 = \frac{1}{2} \langle m(v - V)^2 \rangle$ and $P = \frac{2}{3} \mathcal{E}_0$. We can now use the thermodynamic relation $d\mathcal{E}_0 = T ds + \frac{\mu}{m} d\rho$ to determine the entropy equation. We get

$$\frac{\partial_t (\sqrt{g} s)}{\sqrt{g}} + \nabla_k (s V^k) = 0. \quad (49)$$

This result can also be derived directly from the Boltzmann equation. Computing the moment of the Boltzmann equation with regard to $\log(f)$ (for classical particles,

$\log(f/(1 \pm f))$ for Bose/Fermi statistics) we obtain equ. (3) with the entropy density given by

$$s = \int d\Gamma \sqrt{g} \left[f \log \left(\frac{1+af}{f} \right) + a \log(1+af) \right], \quad (50)$$

and the dissipative function

$$\frac{R}{T} = \int d\Gamma \sqrt{g} \log \left(\frac{f}{1+af} \right) C[f], \quad (51)$$

with $a = 0, \pm 1$ for classical, Bose and Fermi statistics, respectively. In the case of classical statistics equ. (51) implies that R vanishes provided the distribution function is an exponential of the collision invariants. The same result holds for quantum statistics, with the exponential replaced by Bose-Einstein or Fermi-Dirac distribution functions. This implies that

$$f^{(0)}(t, x, p) = f^{(0)} \left(\frac{g_{ij}(t, x) c^i(t, x) c^j(t, x)}{2mT(t, x)} - \alpha(t, x) \right), \quad (52)$$

where $f^{(0)}$ is the Maxwell, Bose-Einstein, or Fermi-Dirac distribution and $\alpha = \mu/T$.

B. First order solution

In this section and the next we will obtain a solution of the Boltzmann equation to second order in the gradients of the hydrodynamic variables. We write the distribution function as

$$f = f^{(0)} + \delta f = f^{(0)} + f^{(1)} + f^{(2)} + \dots, \quad (53)$$

where $f^{(i)}$ contains terms with i gradients. Since we are interested in the structure of the terms that appear in δf , and not in computing the values of transport coefficients for a specific theory, we will consider a very simple choice for the collision term, the relaxation time approximation

$$C[f] = -\frac{\delta f}{\lambda}. \quad (54)$$

Conformal invariance implies that λ must have conformal dimension 1. This is physically reasonable, since λ has units of time. The collision term conserves mass, momentum, and energy. These conservation laws constrain the form of δf . We have

$$\int d\chi_c \delta f = 0, \quad \int d\chi_c \delta f c^i = 0, \quad \int d\chi_c \delta f g_{ij} c^i c^j = 0, \quad (55)$$

where we have defined $d\chi_c = \sqrt{g} d^3c / (2\pi)^3$. The constraints imply that the correction to the stress tensor due to δf is given by

$$\delta\Pi^{ij} = \int d\chi_c \delta f p^i p^j = \int d\chi_c \delta f \left(c^i c^j - \frac{1}{3} g^{ij} c^2 \right) \quad (56)$$

where $c^2 \equiv g_{ij} c^i c^j$ and from now on we will use units $m \equiv 1$. In order to compute $f^{(1)}$ we will follow the procedure of Chapman and Enskog [40, 41] and compute the LHS of the Boltzmann equation using the zero'th order solution equ. (52). We find

$$\begin{aligned} \mathcal{D}f^{(0)} = & -\frac{f^{(0)}(1 + af^{(0)})}{2T} \left(c^i c^j \dot{g}_{ij} + 2\dot{c}^i c^j g_{ij} + p^k c^i c^j \partial_k g_{ij} + 2p^k c^j g_{ij} \partial_k c^i \right. \\ & \left. - 2g_{ik} \Gamma_{jl}^i p^j p^l c^k - 2\dot{g}_{kj} p^j c^k - g_{ij} c^i c^j (\partial_t \log T - p^k q_k) - 2T (\dot{\alpha} + p^k \partial_k \alpha) \right). \end{aligned} \quad (57)$$

Using $\dot{c}_i = -\partial_t V_i$, $\partial_j c_i = -\partial_j V_i$, and $p_i = c_i + V_i$ we obtain

$$\begin{aligned} \mathcal{D}f^{(0)} = & \frac{f^{(0)}(1 + af^{(0)})}{2T} \left\{ (2T\dot{\alpha} + 2TV^k \partial_k \alpha) + 2c^i \left(\dot{V}_i + V^k \nabla_k V_i + T \partial_i \alpha - \frac{a_1 q_i}{2} \right) \right. \\ & + c^2 \left(\partial_t \log T - V^i q_i + \frac{2}{3} \left(\nabla \cdot V + \frac{\dot{g}}{2g} \right) \right) \\ & + \left[c^j c^k - \frac{1}{3} g^{jk} c^2 \right] \left(\nabla_j V_k + \nabla_k V_j + \dot{g}_{kj} - \frac{2}{3} g_{jk} \left(\nabla \cdot V + \frac{\dot{g}}{2g} \right) \right) \\ & \left. - c^2 c^k q_k + a_1 c^k q_k \right\}, \end{aligned} \quad (58)$$

where, in order to make the constraints equ. (55) manifest, we have added and subtracted the term $a_1 c^k q_k$. The first three terms in equ. (58) are proportional to the collision invariants and must vanish by the equations of motion in order to satisfy the constraints. The fourth term, which is proportional to $c_i c_j - \frac{1}{3} c^2 g_{ik}$, automatically satisfies the constraints. The fifth term, proportional to $c^2 c^k$, satisfies the constraints when combined with $a_1 c^k$ term. This implies

$$\int d\chi_c f^{(0)} (1 + af^{(0)}) (c^4 - a_1 c^2) = 0. \quad (59)$$

Using the thermodynamic identities derived in App. A we find $a_1(t, x) = 5P(t, x)/n(t, x)$. For a Maxwell gas the equation of state is $P = nT$ and $a_1 = 5T$. The terms proportional to the collision invariants 1, c_i and c^2 involve time derivatives of the hydrodynamic variables

α, T and V_i . They are easily seen to vanish by the Euler equations

$$0 = \dot{n} + V^k \partial_k n + n \left(\nabla \cdot V + \frac{\dot{g}}{2g} \right), \quad (60)$$

$$0 = \dot{V}_i + V^k \nabla_k V_i + \frac{\partial_i P}{n}, \quad (61)$$

$$0 = \dot{\mathcal{E}}_0 + V^k \partial_k \mathcal{E}_0 + (\mathcal{E}_0 + P) \left(\nabla \cdot V + \frac{\dot{g}}{2g} \right). \quad (62)$$

We can now solve the first order Boltzmann equation $\mathcal{D}f^{(0)} = C[f^{(0)} + f^{(1)}]$ for $f^{(1)}$. Using equ. (54) we find

$$f^{(1)} = -\frac{\lambda f^{(0)}(1 + a f^{(0)})}{2T} (c^i c^j \sigma_{ij} - (c^2 - a_1) c^k q_k), \quad (63)$$

where σ_{ij} is the shear tensor defined in equ. (15). Using the result for $f^{(1)}$ we can compute the dissipative corrections to Π_{ij} and J_s^i . We find $\delta \Pi_{ij} = -\eta \sigma_{ij}$ and $\delta J_s^i = \kappa q^i$. We note that the solution of the Boltzmann equation in curved space automatically leads to a conformal and diffeomorphism invariant stress tensor and entropy current. The shear viscosity and thermal conductivity are given by

$$\eta = \frac{\lambda}{15T} \int d\chi_c f^{(0)} (1 + a f^{(0)}) c^4, \quad (64)$$

$$\kappa = \frac{\lambda}{12T^2} \int d\chi_c f^{(0)} (1 + a f^{(0)}) (c^2 - a_1) c^4. \quad (65)$$

Using the results in App. A we can show that $\eta = \lambda P$ and $\kappa = \frac{\lambda}{12T}(7Q - 75P^2/n)$, where $Q = \langle c^4 \rangle$. In the case of a Maxwell gas the result for the shear viscosity reduces to the familiar form $\eta = \lambda n T$. In this limit we also find $\kappa = \frac{5}{2} \lambda n T$, which corresponds to a Prandtl ratio $Pr = c_p \eta / \kappa = 1$. Note that this result is a consequence of the simple form of the collision term in equ. (54). In a more complete treatment the high temperature limit of the Prandtl ratio is $Pr = \frac{2}{3}$ [42]. Finally, the dissipative function R is given by

$$R = \frac{\eta}{2} \sigma_{ij} \sigma^{ij} + \kappa T q_i q^i, \quad (66)$$

which is manifestly positive for $\lambda \geq 0$.

C. Second order solution

At second order in the derivative expansion the Boltzmann equation reduces to

$$\mathcal{D}f^{(1)} + \left(\mathcal{D}f^{(0)} + \frac{f^{(1)}}{\lambda} \right) = -\frac{f^{(2)}}{\lambda}. \quad (67)$$

We note that $(\mathcal{D}f^{(0)} + \frac{1}{\lambda}f^{(1)})$ vanishes at first order. At second order we have to include gradient terms in the equation of motion and this term is not zero. The Boltzmann operator acting on $f^{(1)}$ gives

$$\begin{aligned} \mathcal{D}f^{(1)} = & -\frac{\lambda f^{(0)}(1 + af^{(0)})}{2T} \left\{ \beta (c^i c^j \sigma_{ij} - (c^2 - a_1) c^k q_k)^2 \right. \\ & + (c^i c^j \sigma_{ij} - (c^2 - a_1) c^k q_k) \left(D \log(\lambda) - D(\log T) \right) \\ & \left. + \mathcal{D} (c^i c^j \sigma_{ij} - (c^2 - a_1) c^k q_k) \right\}, \end{aligned} \quad (68)$$

where we have defined $\beta = (1 + 2af^{(0)})/(2T)$. We will decompose this expression as

$$\mathcal{D}f^{(1)} = (\mathcal{D}f)_{orth} + (\mathcal{D}f)_{scal} + (\mathcal{D}f)_{zm}, \quad (69)$$

where $(\mathcal{D}f)_{orth}$ are terms that are orthogonal to the zero modes of the collision operator, $(\mathcal{D}f)_{scal}$ is orthogonal to the zero modes but does not contribute to the conserved currents Π^{ij} and j_ϵ^i , and $(\mathcal{D}f)_{zm}$ contains terms that are proportional to the zero modes. This means that $(\mathcal{D}f)_{zm}$ must cancel against $(\mathcal{D}f^{(0)} + \frac{1}{\lambda}f^{(1)})$. Before we check this we collect the terms in $(\mathcal{D}f)_{orth}$. We get

$$\begin{aligned} (\mathcal{D}f)_{orth} = & -\frac{\lambda f^{(0)}(1 + af^{(0)})}{2T} \left\{ \beta \left[c^i c^j c^k c^l \sigma_{ij} \sigma_{kl} - \frac{2}{15} c^4 \sigma^2 \right] + a_1^2 \beta \left[c^i c^j q_i q_j - \frac{c^2 q^2}{3} \right] \right. \\ & + \beta \left[c^4 c^i c^j q_i q_j - \frac{c^6 q^2}{3} \right] - 2a_1 \beta \left[c^2 c^i c^j q_i q_j - \frac{c^4 q^2}{3} \right] \\ & + 2a_1 \left[\beta c^i c^j c^k \sigma_{ij} q_k - c^i q_k \sigma_i^k \right] - 2 \left[\beta c^2 c^i c^j c^k \sigma_{ij} q_k - \frac{7a_1}{5} c^i q_k \sigma_i^k \right] \\ & + [c^i c^j \sigma_{ij} - c^2 c^i q_i + a_1 c^i q_i] (D \log(\lambda) - \partial_t \log T + V^k q_k) \\ & + \left[c^i c^j c^k \sigma_{ij} q_k - \frac{2a_1}{5} c^i q_k \sigma_i^k \right] - \left[c^2 c^i c^j q_i q_j - \frac{c^4 q^2}{3} \right] + \left[a_1 c^i c^j q_i q_j - \frac{a_1 c^2 q^2}{3} \right] \\ & + \left[c^i c^j c^k - \frac{2a_1}{5} c^i g^{jk} \right] (\nabla_k \sigma_{ij} + \sigma_{ij} q_k) - \left[c^2 c^i c^j - \frac{g^{ij} c^4}{3} \right] \nabla_i q_j \\ & + [c^2 c^k - a_1 c^k] \left(q^i \nabla_k V_i + \dot{g}_{ik} q^i - \dot{q}_k - V^i \nabla_i q_k + \frac{2\langle \sigma \rangle}{3} q_k \right) \\ & + \left[c^i c^j - \frac{g^{ij} c^2}{3} \right] \left(g_{jk} \dot{\sigma}_i^k + V^k \nabla_k \sigma_{ij} + a_1 \nabla_i q_j - \sigma_{ik} \sigma_j^k - \Omega_{ik} \sigma_j^k - \frac{2\sigma_{ij} \langle \sigma \rangle}{3} \right. \\ & \left. + 2 \left(\dot{V}_i + V^k \nabla_k V_i \right) q_j + q_i \partial_j a_1 \right) \left. \right\}. \end{aligned} \quad (70)$$

The terms in the square brackets are automatically orthogonal to the zero modes. As in the previous section this is achieved by adding and subtracting terms. Subtraction terms

that are proportional to the zero modes are collected in $(\mathcal{D}f)_{zm}$. The remaining terms are collected in $(\mathcal{D}f)_{scal}$. We have

$$\begin{aligned}
(\mathcal{D}f)_{scal} = & -\frac{\lambda f^{(0)}(1 + af^{(0)})}{2T} \left\{ \frac{2\beta\sigma^2}{15} [c^4 - b_{41}c^2 - b_{40}] + \frac{\beta a_1^2 q^2}{3} [c^2 - b_{21}c^2 - b_{20}] \right. \\
& + \frac{\beta q^2}{3} [c^6 - b_{61}c^2 - b_{60}] - \frac{2\beta a_1 q^2}{3} [c^4 - b_{41}c^2 - b_{40}] \\
& \left. - \frac{q^2}{3} [c^4 - d_1c^2 - d_0] - \frac{\nabla_i q^i}{3} [c^4 - d_1c^2 - d_0] \right\}. \tag{71}
\end{aligned}$$

The constants $b_{20}, b_{21}, b_{40}, b_{41}, b_{60}, b_{61}$ and d_1, d_0 are analogous to the constant a_1 defined in equ. (59). We derive the relevant orthogonality relations in App. B. Finally, $(\mathcal{D}f)_{zm}$ is given by

$$\begin{aligned}
(\mathcal{D}f)_{zm} = & -\frac{\lambda f^{(0)}(1 + af^{(0)})}{2T} \left\{ a_1 T q^i \partial_i \alpha + \frac{2b_{40}}{15} \sigma^2 + \frac{7d_0}{6} q^2 - \frac{d_0}{3} \nabla_i q^i \right. \\
& + c^2 \left(\frac{\theta}{3n} \right) \left[-a_1 T q^i \partial_i \alpha - \frac{2b_{40}}{15} \sigma^2 - \frac{7d_0}{6} q^2 + \frac{d_0}{3} \nabla_i q^i \right] \\
& \left. + 2c^i \left[\frac{\sigma_i^j \partial_j P}{n} + \frac{P}{n} \nabla_k \sigma_i^k \right] \right\}, \tag{72}
\end{aligned}$$

where we have used the Euler equations to eliminate time derivatives of the hydrodynamic variables. This is consistent at this order in the derivative expansion. The parameter θ is defined in App. A (for a Maxwell gas $\theta = n/T$). We now use the Navier-Stokes equations

$$\begin{aligned}
0 = & 2T\dot{\alpha} + 2TV^k \partial_k \alpha \\
& + \lambda \left[a_1 T q^i \partial_i \alpha + \frac{2b_{40}}{15} \sigma^2 + \frac{7d_0}{6} q^2 - \frac{d_0}{3} \nabla_i q^i \right], \tag{73}
\end{aligned}$$

$$0 = \dot{V}_i + V^k \nabla_k V_i + T \partial_i \alpha - \frac{a_1 q_i}{2} + \lambda \left[\frac{\sigma_i^j \partial_j P}{n} + \frac{P}{n} \nabla_k \sigma_i^k \right], \tag{74}$$

$$\begin{aligned}
0 = & \partial_t \log T - V^i q_i + \frac{2}{3} \left(\nabla \cdot V + \frac{\dot{g}}{2g} \right) \\
& + \lambda \left(\frac{\theta}{3n} \right) \left[-a_1 T q^i \partial_i \alpha - \frac{2b_{40}}{15} \sigma^2 - \frac{7d_0}{6} q^2 + \frac{d_0}{3} \nabla_i q^i \right], \tag{75}
\end{aligned}$$

to confirm that $(\mathcal{D}f)_{zm} + (\mathcal{D}f^{(0)} + \frac{1}{\lambda} f^{(1)}) = 0$. Solving equ. (67) for $f^{(2)}$ we get

$$f^{(2)} = -\lambda [(\mathcal{D}f)_{orth} + (\mathcal{D}f)_{scal}]. \tag{76}$$

where $(\mathcal{D}f)_{orth}$ and $(\mathcal{D}f)_{scal}$ are defined in equ. (70) and (71). We note that $(\mathcal{D}f)_{orth}$ depends on $D \log(\lambda)$. This term is necessary in order to respect conformal invariance. Dimensional analysis implies that $\lambda = T^{-1} h(\alpha)$ for some undetermined function h . Using the Euler equation we can then show that $D \log(\lambda) = \frac{2}{3} \langle \sigma \rangle$.

D. Stress tensor and entropy current at second order

Using the result for $f^{(2)}$ obtained in the previous section we can compute the dissipative correction to the stress tensor at second order in the gradient expansion. As discussed above only $(\mathcal{D}f)_{orth}$ contributes to the conserved currents. We will write $f_{orth}^{(2)} = -\lambda(\mathcal{D}f)_{orth}$. We can simplify $f_{orth}^{(2)}$ by combining some terms, and by using the Euler equation. We find

$$\begin{aligned}
f_{orth}^{(2)} = & \frac{\lambda^2 f^{(0)}(1 + af^{(0)})}{2T} \left\{ \beta \left[c^i c^j c^k c^l \sigma_{ij} \sigma_{kl} - \frac{2}{15} c^4 \sigma^2 \right] \right. \\
& + (c^2 - a_1) (\beta c^2 - a_1 \beta - 1) \left[c^i c^j q_i q_j - \frac{c^2 q^2}{3} \right] \\
& + [c^i c^j \sigma_{ij} - c^2 c^i q_i + a_1 c^i q_i] \frac{2}{3} \langle \sigma \rangle + 2 (a_1 \beta + 1 - \beta c^2) c^i c^j c^k \sigma_{ij} q_k \\
& + \left[c^i c^j c^k - \frac{2a_1}{5} c^i g^{jk} \right] \nabla_k \sigma_{ij} + [c^2 c^k - a_1 c^k] (q^i \nabla_k V_i + \dot{g}_{ik} q^i - \dot{q}_k - V^i \nabla_i q_k) \\
& + \left[c^i c^j - \frac{g^{ij} c^2}{3} \right] \left(g_{jk} \dot{\sigma}_i^k + V^k \nabla_k \sigma_{ij} - \sigma_{ik} \dot{\sigma}_j^k - \Omega_{ik} \sigma_j^k \right. \\
& \left. \left. + \frac{(3n^2 - 5P\theta)T}{n^2} q_i \partial_j \alpha - (c^2 - a_1) \nabla_i q_j \right) \right\}. \tag{77}
\end{aligned}$$

The stress tensor can be determined using equ. (56). The result is

$$\begin{aligned}
\delta \Pi_{(2)}^{ij} = & \lambda \eta \left(\sigma^{ik} \sigma_k^j - \frac{g^{ij}}{3} \sigma^2 \right) - \frac{4\lambda^2}{5} \nabla^{(i} \left(\frac{\kappa T}{\lambda} q^{j)} \right) + \frac{4\lambda^2 g^{ij}}{15} \nabla_k \left(\frac{\kappa T}{\lambda} q^k \right) \\
& + \lambda \eta \left(g^{jk} \dot{\sigma}_k^i + V^k \nabla_k \sigma^{ij} + \frac{2}{3} \langle \sigma \rangle \sigma^{ij} - \Omega^{(ik} \sigma_k^{j)} \right). \tag{78}
\end{aligned}$$

This result is in agreement with the general form of $\delta \Pi^{ij}$ derived in Sect. II E. We note that some of the terms allowed by symmetry vanish in the kinetic theory calculation. This includes the $\Omega^i_k \Omega^{jk}$ structure and the $\nabla_i P \nabla_j P$ and $\nabla_i \nabla_j P$ terms. The kinetic theory also does not give a pure curvature term proportional to R_{ij} . We also note that the coefficient of the $g^{jk} \dot{\sigma}_k^i$ term can be written as $\tau_R \eta$, where τ_R is the relaxation time for the dissipative stresses. Our result shows that $\tau_R = \lambda = \eta/P$. This result is in agreement with the calculation of spectral function of the stress tensor in [43], and with previous work on relaxation effects in dilute Bose and Fermi gases [44, 45].

Finally, we compute dissipative corrections to the entropy current and the dissipative function. We will use the expression for j_s^i and R that are obtained by taking moments of the Boltzmann equation with $\log(f/(1 + af))$, see Sect. III A. As a consequence of the H-theorem these equations automatically satisfy the second law of thermodynamics. This is

no longer the case if the result is expanded in derivatives of the thermodynamic variables. The distribution function at second order in the gradient expansion determines R to third order in gradients, and this expression is not manifestly positive. It is possible, however, to add certain fourth order terms and obtain a manifestly positive result. This is the strategy we have adopted. We find

$$j_s^i = \kappa q^i - \lambda \left(\frac{27\kappa q_k \sigma^{ki}}{10} - \frac{2\kappa}{5} \nabla_k \sigma^{ki} - \frac{\kappa q_k \Omega^{ik}}{2} + \frac{\kappa q^i \langle \sigma \rangle}{3} + \kappa \dot{q}^i + \frac{\kappa g^{ik} \dot{g}_{kj} q^j}{2} + \kappa V^k \nabla_k q^i \right), \quad (79)$$

and

$$\begin{aligned} \frac{R}{T} = & \frac{\eta}{2} \left(\sigma - \lambda \left[\frac{1}{2} \sigma \sigma - \sigma \Omega + \frac{2}{3} \langle \sigma \rangle \sigma + \dot{\sigma} + V^k \nabla_k \sigma \right] \right)^2 \\ & + \kappa T \left(q - \lambda \left[\frac{27q\sigma}{20} - \frac{q\Omega}{2} + \frac{q\langle \sigma \rangle}{3} + \dot{q} + V^k \nabla_k q \right] \right)^2. \end{aligned} \quad (80)$$

In equ. (80) we have suppressed tensor indices. Contractions are defined as $\sigma\sigma = \sigma_{ik}\sigma_j^k$, $\sigma\Omega = \sigma_{ik}\Omega_j^k$ etc., and $(\sigma)^2 = \sigma_{ij}\sigma^{ij}$. We note that the dissipative function is manifestly positive provided the first order relations $\eta \geq 0$ and $\kappa \geq 0$ are satisfied, and that the entropy current is of the form discussed in Sect. II F.

IV. OUTLOOK AND CONCLUSIONS

In this work we have studied the constraints imposed by conformal invariance on the form of the hydrodynamic equations at second order in the gradient expansion. We find that the most general form of the stress tensor is given by

$$\begin{aligned} \delta\Pi_{ij} = & -\eta\sigma_{ij} + \eta\tau_R \left(g_{ik}\dot{\sigma}_j^k + V^k \nabla_k \sigma_{ij} + \frac{2}{3} \langle \sigma \rangle \sigma_{ij} \right) + \lambda_1 \sigma_{\langle i}^{k} \sigma_{j\rangle k} + \lambda_2 \sigma_{\langle i}^{k} \Omega_{j\rangle k} \\ & + \lambda_3 \Omega_{\langle i}^{k} \Omega_{j\rangle k} + \gamma_1 \nabla_{\langle i} T \nabla_{j\rangle} T + \gamma_2 \nabla_{\langle i} P \nabla_{j\rangle} P + \gamma_3 \nabla_{\langle i} T \nabla_{j\rangle} P \\ & + \gamma_4 \nabla_{\langle i} \nabla_{j\rangle} T + \gamma_5 \nabla_{\langle i} \nabla_{j\rangle} P + \kappa_R R_{\langle ij \rangle}, \end{aligned} \quad (81)$$

where we have suppressed terms involving the gauge fields. Conformal symmetry constrains the form of the comoving derivative of σ_{ij} and eliminates possible trace terms like $g_{ij}\sigma_{kl}\sigma^{kl}$. For phenomenological applications it is useful to rewrite the second order equations as a relaxation equation for the viscous stress $\pi_{ij} \equiv \delta\Pi_{ij}$. For this purpose we use the first order

relation $\pi_{ij} = -\eta\sigma_{ij}$ and rewrite equ. (81) as

$$\pi_{ij} = -\eta\sigma_{ij} - \tau_R \left(g_{ik} \dot{\pi}_j^k + V^k \nabla_k \pi_{ij} + \frac{5}{3} \langle \sigma \rangle \pi_{ij} \right) + \lambda_1 \pi_{\langle i}^k \pi_{j \rangle k} + \dots, \quad (82)$$

where \dots refers to the terms proportional to $\lambda_{2,3}, \gamma_i$ and κ_R . In deriving equ. (82) we have also used $\eta(n, T) = nf(\alpha)$, where f is a function of α , and the Euler equations for n and α . Equ. (82) has the same structure as the second order equations considered by Israel and Stewart [29], but the coefficient $\frac{5}{3}$ is specific to the non-relativistic conformal case.

We have checked the conformal constraints by computing the dissipative contribution to the stress tensor, the entropy current, and the dissipative function in kinetic theory. We find that $\delta\Pi_{ij}$ is indeed of the form given in equ. (81), but that some terms allowed by the symmetries do not appear in kinetic theory. The relaxation time for the dissipative stresses is given by $\tau_R = \eta/P$. The entropy current and the dissipative function also satisfy the conformal constraints. As in the case of the stress tensor, not all possible terms appear. The relaxation time for the entropy current is equal to the viscous relaxation time.

There are a number of issues that we have not addressed in this paper. We have studied the constraints that arise from conformal symmetry and Galilean invariance, but we have not investigated the conditions that arise from the second law of thermodynamics. These conditions should have the form of inequalities for the transport coefficients, analogous to the first order relations $\eta \geq 0$ and $\kappa \geq 0$. In the kinetic theory calculation the second law is satisfied automatically, but in the simple model considered here the transport coefficients are all governed by a single relaxation time λ . We have also not attempted to derive Kubo relations for the coefficients τ_R , λ_i , and γ_i . The analogous Kubo formulas in the relativistic case were obtained in [32]. Finally, it would be interesting to compute the spectral functions for the stress tensor and the energy current at second order in the derivative expansion. These results would be useful in connection with attempts to compute transport coefficients using Quantum Monte Carlo methods [46].

Acknowledgments: This work was supported in parts by the US Department of Energy grant DE-FG02-03ER41260. We thank Dam Son for a useful discussion.

Appendix A: Thermodynamic relations

In this appendix we collect a number of thermodynamic relations. In kinetic theory the number density and the pressure are determined by equ. (43) and (44). We have

$$n(T, \alpha) = \int d\chi_c f^{(0)} \left(\frac{c^2}{2T} - \alpha \right), \quad (\text{A1})$$

$$P(T, \alpha) = \frac{1}{3} \int d\chi_c c^2 f^{(0)} \left(\frac{c^2}{2T} - \alpha \right), \quad (\text{A2})$$

where $d\chi_c$ is defined below equ. (55) and we have set $m = 1$. We also define

$$Q(T, \alpha) = \int d\chi_c c^4 f^{(0)} \left(\frac{c^2}{2T} - \alpha \right). \quad (\text{A3})$$

We can now compute various differentials. Differentials of n are given by

$$\left. \frac{\partial n(T, \alpha)}{\partial T} \right|_{\alpha} = \partial_T \left\{ \int d\chi_c f^{(0)} \left(\frac{c^2}{2T} - \alpha \right) \right\} = \frac{1}{2T^2} \int d\chi_c c^2 f^{(0)} (1 + a f^{(0)}) = \frac{3n}{2T}, \quad (\text{A4})$$

$$\left. \frac{\partial n(T, \alpha)}{\partial \alpha} \right|_T = \partial_{\alpha} \left\{ \int d\chi_c f^{(0)} \left(\frac{c^2}{2T} - \alpha \right) \right\} = \int d\chi_c f^{(0)} (1 + a f^{(0)}) = \theta T, \quad (\text{A5})$$

where $a = 0, \pm 1$ corresponds to Maxwell, Bose-Einstein and Fermi-Dirac distribution functions, respectively. From the above relations we find

$$dn = \theta T d\alpha + \frac{3n}{2} d(\log T). \quad (\text{A6})$$

Differentials of the pressure can be computed in the same way. We get

$$\left. \frac{\partial P(T, \alpha)}{\partial T} \right|_{\alpha} = \frac{1}{3} \partial_T \left\{ \int d\chi_c c^2 f^{(0)} \left(\frac{c^2}{2T} - \alpha \right) \right\} = \frac{5P}{2T}, \quad (\text{A7})$$

$$\left. \frac{\partial P(T, \alpha)}{\partial \alpha} \right|_T = \frac{1}{3} \partial_{\alpha} \left\{ \int d\chi_c c^2 f^{(0)} \left(\frac{c^2}{2T} - \alpha \right) \right\} = nT, \quad (\text{A8})$$

and

$$dP = nT d\alpha + \frac{5P}{2} d(\log T), \quad d\mathcal{E}_0 = \frac{3nT}{2} d\alpha + \frac{15P}{4} d(\log T). \quad (\text{A9})$$

From $\mathcal{E}_0 + P = \mu n + Ts$ we obtain

$$\left. \frac{\partial s(T, \alpha)}{\partial T} \right|_{\alpha} = -\frac{5P}{2nT^2} + \frac{5}{2nT} \left. \frac{\partial P(T, \alpha)}{\partial T} \right|_{\alpha} - \frac{5P}{2n^2 T} \left. \frac{\partial n(T, \alpha)}{\partial T} \right|_{\alpha} = 0, \quad (\text{A10})$$

$$\left. \frac{\partial s(T, \alpha)}{\partial \alpha} \right|_T = -1 + \frac{5}{2nT} \left. \frac{\partial P(T, \alpha)}{\partial \alpha} \right|_T - \frac{5P}{2n^2 T} \left. \frac{\partial n(T, \alpha)}{\partial \alpha} \right|_T = \frac{3n^2 - 5P\theta}{2n^2}, \quad (\text{A11})$$

and

$$ds = \frac{3n^2 - 5P\theta}{2n^2} d\alpha. \quad (\text{A12})$$

For a Maxwell gas this relation becomes $ds = -d\alpha$. The differential of Q is given by

$$dQ = 15PT d\alpha + \frac{7Q}{2} d(\log T). \quad (\text{A13})$$

Appendix B: Orthogonality relations

In this appendix we collect the orthogonality relations that determine the coefficients $b_{20}, b_{21}, b_{40}, b_{41}, b_{60}, b_{61}$ and d_1, d_0 introduced in Sec. III C. The coefficients b_{20}, b_{21} are defined by

$$0 = \int d\chi_c f^{(0)}(1 + af^{(0)}) \left(\frac{1 + 2af^{(0)}}{2T} c^2 - b_{21}c^2 - b_{20} \right), \quad (\text{B1})$$

$$0 = \int d\chi_c f^{(0)}(1 + af^{(0)}) \left(\frac{1 + 2af^{(0)}}{2T} c^4 - b_{21}c^4 - b_{20}c^2 \right). \quad (\text{B2})$$

These relations imply

$$b_{21} = \frac{-n\theta}{3n^2 - 5P\theta}, \quad b_{20} = \frac{15(n^2 - P\theta)}{2(3n^2 - 5P\theta)}. \quad (\text{B3})$$

The remaining coefficients are determined analogously. We have

$$0 = \int d\chi_c f^{(0)}(1 + af^{(0)}) \left(\frac{1 + 2af^{(0)}}{2T} c^4 - b_{41}c^2 - b_{40} \right), \quad (\text{B4})$$

$$0 = \int d\chi_c f^{(0)}(1 + af^{(0)}) \left(\frac{1 + 2af^{(0)}}{2T} c^6 - b_{41}c^4 - b_{40}c^2 \right), \quad (\text{B5})$$

$$\Rightarrow b_{41} = \frac{5(3n^2 - 7P\theta)}{2(3n^2 - 5P\theta)}, \quad b_{40} = \frac{15Pn}{3n^2 - 5P\theta}, \quad (\text{B6})$$

$$0 = \int d\chi_c f^{(0)}(1 + af^{(0)}) \left(\frac{1 + 2af^{(0)}}{2T} c^6 - b_{61}c^2 - b_{60} \right), \quad (\text{B7})$$

$$0 = \int d\chi_c f^{(0)}(1 + af^{(0)}) \left(\frac{1 + 2af^{(0)}}{2T} c^8 - b_{61}c^4 - b_{60}c^2 \right), \quad (\text{B8})$$

$$\Rightarrow b_{61} = \frac{21(5nP - Q\theta)}{2(3n^2 - 5P\theta)}, \quad b_{60} = \frac{21(3nQ - 25P^2)}{2(3n^2 - 5P\theta)}, \quad (\text{B9})$$

$$0 = \int d\chi_c f^{(0)}(1 + af^{(0)}) (c^4 - d_1c^2 - d_0), \quad (\text{B10})$$

$$0 = \int d\chi_c f^{(0)}(1 + af^{(0)}) (c^6 - d_1c^4 - d_0c^2), \quad (\text{B11})$$

$$\Rightarrow d_1 = \frac{45nP - 7Q\theta}{3(3n^2 - 5P\theta)}, \quad d_0 = \frac{7nQ - 75P^2}{3n^2 - 5P\theta}. \quad (\text{B12})$$

where the thermodynamic variables θ, n, P, Q are defined in the previous section.

[1] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2**, 231 (1998) [arXiv:hep-th/9711200].

- [2] P. Kovtun, D. T. Son and A. O. Starinets, “Holography and hydrodynamics: Diffusion on stretched horizons,” JHEP **0310**, 064 (2003) [arXiv:hep-th/0309213].
- [3] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” JHEP **0802**, 045 (2008) [arXiv:0712.2456 [hep-th]].
- [4] M. Rangamani, “Gravity and Hydrodynamics: Lectures on the fluid-gravity correspondence,” Class. Quant. Grav. **26**, 224003 (2009) [arXiv:0905.4352 [hep-th]].
- [5] R. Baier, P. Romatschke, D. T. Son *et al.*, “Relativistic viscous hydrodynamics, conformal invariance, and holography,” JHEP **0804**, 100 (2008). [arXiv:0712.2451 [hep-th]].
- [6] I. Kanitscheider, K. Skenderis, “Universal hydrodynamics of non-conformal branes,” JHEP **0904**, 062 (2009) [arXiv:0901.1487 [hep-th]].
- [7] C. P. Herzog, N. Lisker, P. Surowka, A. Yarom, “Transport in holographic superfluids,” [arXiv:1101.3330 [hep-th]].
- [8] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, A. Yarom, “A Theory of first order dissipative superfluid dynamics,” [arXiv:1105.3733 [hep-th]].
- [9] I. Fouxon, Y. Oz, “Conformal Field Theory as Microscopic Dynamics of Incompressible Euler and Navier-Stokes Equations,” Phys. Rev. Lett. **101**, 261602 (2008). [arXiv:0809.4512 [hep-th]].
- [10] T. Schäfer and D. Teaney, “Nearly Perfect Fluidity: From Cold Atomic Gases to Hot Quark Gluon Plasmas,” Rept. Prog. Phys. **72**, 126001 (2009) [arXiv:0904.3107 [hep-ph]].
- [11] S. S. Adler *et al.* [PHENIX Collaboration], “Elliptic flow of identified hadrons in Au + Au collisions at $\sqrt{s(NN)} = 200$ -GeV,” Phys. Rev. Lett. **91**, 182301 (2003) [arXiv:nucl-ex/0305013].
- [12] K. M. O’Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, J. E. Thomas, “Observation of a Strongly-Interacting Degenerate Fermi Gas of Atoms” Science **298**, 2179 (2002) [cond-mat/0212463].
- [13] K. Dusling and D. Teaney, “Simulating elliptic flow with viscous hydrodynamics,” Phys. Rev. C **77**, 034905 (2008) [arXiv:0710.5932 [nucl-th]].
- [14] P. Romatschke and U. Romatschke, “Viscosity Information from Relativistic Nuclear Collisions: How Perfect is the Fluid Observed at RHIC?,” Phys. Rev. Lett. **99**, 172301 (2007) [arXiv:0706.1522 [nucl-th]].
- [15] H. Song and U. W. Heinz, “Causal viscous hydrodynamics in 2+1 dimensions for relativistic

- heavy-ion collisions,” *Phys. Rev. C* **77**, 064901 (2008) [arXiv:0712.3715 [nucl-th]].
- [16] T. Schäfer, “The Shear Viscosity to Entropy Density Ratio of Trapped Fermions in the Unitarity Limit,” *Phys. Rev. A* **76**, 063618 (2007) [arXiv:cond-mat/0701251].
 - [17] A. Turlapov, J. Kinast, B. Clancy, L. Luo, J. Joseph, J. E. Thomas, “Is a Gas of Strongly Interacting Atomic Fermions a Nearly Perfect Fluid” *J. Low Temp. Phys.* **150**, 567 (2008) [arXiv:0707.2574].
 - [18] T. Schäfer and C. Chafin, “Scaling Flows and Dissipation in the Dilute Fermi Gas at Unitarity,” in “The BCS-BEC Crossover and the Unitary Fermi Gas” Springer Lecture Notes in Physics, Vol. 836, Zwerger, Wilhelm (Ed.) (2011) [arXiv:0912.4236 [cond-mat.quant-gas]].
 - [19] C. Cao, E. Elliott, J. Joseph, H. Wu, J. Petricka, T. Schäfer, J. E. Thomas, “Observation of Universal Temperature Scaling in the Quantum Viscosity of a Unitary Fermi Gas,” *Science* **331**, 58 (2011) [arXiv:1007.2625 [cond-mat.quant-gas]].
 - [20] G. Policastro, D. T. Son, A. O. Starinets, “The Shear viscosity of strongly coupled $N=4$ susy Yang-Mills plasma,” *Phys. Rev. Lett.* **87**, 081601 (2001). [hep-th/0104066].
 - [21] P. Kovtun, D. T. Son and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” *Phys. Rev. Lett.* **94**, 111601 (2005) [arXiv:hep-th/0405231].
 - [22] D. T. Son and M. Wingate, “General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas,” *Annals Phys.* **321**, 197 (2006) [arXiv:cond-mat/0509786].
 - [23] T. Mehen, I. W. Stewart, M. B. Wise, “Conformal invariance for nonrelativistic field theory,” *Phys. Lett.* **B474**, 145-152 (2000). [hep-th/9910025].
 - [24] C. R. Hagen, “Scale and conformal transformations in galilean-covariant field theory,” *Phys. Rev.* **D5**, 377-388 (1972).
 - [25] D. T. Son, “Vanishing bulk viscosities and conformal inv of unitary Fermi gas,” *Phys. Rev. Lett.* **98**, 020604 (2007) [arXiv:cond-mat/0511721].
 - [26] T. Enss, R. Haussmann, W. Zwerger, “Viscosity and scale invariance in the unitary Fermi gas,” *Annals Phys.* **326**, 770-796 (2011). [arXiv:1008.0007 [cond-mat.quant-gas]].
 - [27] Y. Castin, F. Werner, “The Unitary Gas and its Symmetry Properties” in “The BCS-BEC Crossover and the Unitary Fermi Gas” Springer Lecture Notes in Physics, Vol. 836, Zwerger, Wilhelm (Ed.) (2011) [arXiv:1103.2851 [cond-mat.quant-gas]].
 - [28] D. Burnett, “The distribution of velocities in a slightly non-uniform gas, *Proc. Lond. Math.*

- Soc. **39** 385 (1935).
- [29] W. Israel and J. M. Stewart, “Transient relativistic thermodynamics and kinetic theory,” *Annals Phys.* **118**, 341 (1979).
 - [30] L. S. Garcia-Colina, R. M. Velascoa, F. J. Uribea, “Beyond the Navier-Stokes equations: Burnett hydrodynamics” *Phys. Rep.* **465** 149 (2008).
 - [31] P. Romatschke, “New Developments in Relativistic Viscous Hydrodynamics,” *Int. J. Mod. Phys.* **E19**, 1-53 (2010). [arXiv:0902.3663 [hep-ph]].
 - [32] G. D. Moore, K. A. Sohrabi, “Kubo Formulae for Second-Order Hydrodynamic Coefficients,” *Phys. Rev. Lett.* **106**, 122302 (2011). [arXiv:1007.5333 [hep-ph]].
 - [33] D. T. Son, “Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry,” *Phys. Rev. D* **78**, 046003 (2008) [arXiv:0804.3972 [hep-th]].
 - [34] K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,” *Phys. Rev. Lett.* **101**, 061601 (2008) [arXiv:0804.4053 [hep-th]].
 - [35] C. P. Herzog, M. Rangamani and S. F. Ross, “Heating up Galilean holography,” *JHEP* **0811**, 080 (2008) [arXiv:0807.1099 [hep-th]].
 - [36] L. D. Landau, E. M. Lifshitz, “Fluid Mechanics”, *Course of Theoretical Physics*, Vol.VI, Pergamon Press (1987).
 - [37] H. Andreasson, “The Einstein-Vlasov system / kinetic theory,” *Liv. Rev. Relativity* **8** (2005) [gr-qc/0206069].
 - [38] C. Cercignani, G. M. Kremer, “The Relativistic Boltzmann Equation: Theory and Applications”, Birkhäuser Verlag (2002).
 - [39] Y. Choquet-Bruhat, “General Relativity and the Einstein Equations”, Oxford University Press (2009).
 - [40] S. Chapman and T. G. Cowling, “The Mathematical Theory of Non-Uniform Gases”, Cambridge University Press, 3rd ed. (1970).
 - [41] L. D. Landau, E. M. Lifshitz, “Physical Kinetics”, *Course of Theoretical Physics*, Vol.X, Pergamon Press (1981).
 - [42] M. Braby, J. Chao, T. Schäfer, “Thermal Conductivity and Sound Attenuation in Dilute Atomic Fermi Gases,” *Phys. Rev.* **A82**, 033619 (2010). [arXiv:1003.2601 [cond-mat.quant-gas]].
 - [43] J. Chao, M. Braby, T. Schäfer, “Viscosity spectral functions of the dilute Fermi gas in kinetic

- theory,” *New J. Phys.* **13**, 035014 (2011). [arXiv:1012.0219 [cond-mat.quant-gas]].
- [44] T. Nikuni, A. Griffin “Frequency and damping of hydrodynamic modes in a Bose-condensed gas,” *Phys. Rev. A* **69**, 023604 (2004) [arXiv:cond-mat/0309269].
- [45] G. M. Bruun, H. Smith, “Frequency and damping of the Scissors Mode of a Fermi gas,” *Phys. Rev. A* **76**, 045602 (2007) [arXiv:0709.1617].
- [46] H. B. Meyer, “Transport Properties of the Quark-Gluon Plasma: A Lattice QCD Perspective,” *Eur. Phys. J.* **A47**, 86 (2011) [arXiv:1104.3708 [hep-lat]].